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# Sufficient conditions for the existence of periodic solutions to some second order differential equations with a deviating argument ☆

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**Abstract**

By means of Mawhin's continuation theorem, we study some second order differential equations with a deviating argument:

$$x''(t) = f\big(t, x(t), x\big(t - \tau(t)\big), x'(t)\big) + e(t).$$

Some new results on the existence of periodic solutions are obtained. The interest is that we allow the degree with respect to the variables $x_0, x_1, x_2$ of $f(t, x_0, x_1, x_2)$ to be greater than 1; and also the result (Theorem 3.2) is related to the deviating argument $\tau(t)$. Meanwhile, we give an example to demonstrate our result.


*Keywords:* Periodic solution; Mawhin's continuation theorem; Deviating argument

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## 1. Introduction

In recent years, some researchers used Mawhin's continuation theorem to study the existence of periodic solutions to some second order differential equations with a deviating argument [1–7]. For example, Ma studied a kind of delay Duffing equation of the type [2]

$$x''(t) + m^2 x(t) + g\big(x(t-\tau)\big) = p(t). \tag{1.1}$$

He established several criteria to guarantee the existence of periodic solutions of Eq. (1.1) by assuming

$$M = \sup_{x\in R}\big|g(x)\big| < \infty. \tag{1.2}$$

In [3], we discussed the existence of periodic solutions for some second order differential equations with two deviating arguments:

$$x''(t) = f\big(t, x\big(t-\tau_0(t)\big)\big)x'(t) + \beta(t)g\big(x\big(t-\tau(t)\big)\big) + p(t). \tag{1.3}$$

However, the growth condition imposed on $g(x)$ is

$$\lim_{|x|\to+\infty}\sup\frac{|g(x)|}{|x|} \leqslant r, \tag{1.4}$$

where $r \geqslant 0$ is a constant. In [6], we studied some Rayleigh equations with a deviating argument

$$x''(t) + f\big(x'(t)\big) + g\big(x\big(t-\tau(t)\big)\big) = p(t). \tag{1.5}$$

The condition imposed on $g(x)$ is in the following form:

$$\big|g(x) - g(y)\big| \leqslant L|x-y|, \quad \forall x, y \in R, \tag{1.6}$$

where $L > 0$ is a constant.

In present paper, we continue to study the existence of periodic solutions to some differential equations with a deviating argument in the following form:

$$x''(t) = f\big(t, x(t), x\big(t-\tau(t)\big), x'(t)\big) + e(t), \tag{1.7}$$

where $f \in C(R^4, R)$ with $f(t,c,c,0) + e(t) \not\equiv 0$ for each $c \in R$ and $f(t+T, x_0, x_1, x_2) \equiv f(t, x_0, x_1, x_2)$ for $\forall(x_0, x_1, x_2) \in R^3$, $\tau$ and $e \in C(R,R)$ with $\tau(t+T) \equiv \tau(t)$ and $e(t+T) \equiv e(t)$. Clearly, Eqs. (1.1), (1.3), and (1.5) are all special cases of Eq. (1.7). By using Mawhin's continuation theorem, we obtain some new results. The interest is that we allow the degree with respect to the variables $x_0, x_1, x_2$ of $f(t, x_0, x_1, x_2)$ to be greater than 1; and also the result (Theorem 3.1) is related to the deviating argument $\tau(t)$. Meanwhile, we give an example to demonstrate our result. Even if Eq. (1.7) is reduced to the special case of Eqs. (1.3) or (1.5), the conditions imposed on $g(x)$ are different from (1.2), (1.4), and (1.6).

## 2. Main lemmas

The following lemma is crucial for us to investigate the relation between the existence of periodic solutions to Eq. (1.7) and the deviating argument $\tau(t)$.

**Lemma 2.1.** *Let $n_1 > 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R, R)$ with $s(t + T) \equiv s(t)$, and $s(t) \in [-\alpha, 0]$, $\forall t \in [0, T]$ or $s(t) \in [0, \alpha]$, $\forall t \in [0, T]$. Then $\forall x \in C^1(R, R)$ with $x(t + T) \equiv x(t)$, we have*

$$\int_0^T \left|x(t) - x\bigl(t - s(t)\bigr)\right|^{n_1} dt \leqslant \alpha^{n_1} \int_0^T \left|x'(t)\right|^{n_1} dt. \tag{2.1}$$

**Proof.** 1. If $s(t) \in [0, \alpha]$, $\forall t \in [0, T]$, then $\forall t \in [0, T]$. By using Hölder inequality, we have

$$\begin{aligned}\left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} &\leqslant \left( \int_{t-s(t)}^{t} \left|x'(\sigma)\right| d\sigma \right)^{n_1} \leqslant \left|s(t)\right|^{\frac{n_1}{m}} \int_{t-s(t)}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma \\ &\leqslant \alpha^{\frac{n_1}{m}} \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma,\end{aligned}$$

where $m > 1$ is a constant with $\frac{1}{m} + \frac{1}{n_1} = 1$. It follows that

$$\int_0^T \left|x(t) - x\bigl(t - s(t)\bigr)\right|^{n_1} dt = \int_0^T \left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} dt \leqslant \alpha^{\frac{n_1}{m}} \int_0^T \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt. \tag{2.2}$$

In what follows, we will prove in two cases that formula (2.1) holds respectively.

**Case 1.** $\alpha \in [0, T]$. Interchanging the order of $\int_0^T \int_{t-\alpha}^t |x'(\sigma)|^{n_1} d\sigma\, dt$, we get

$$\begin{aligned}&\int_0^T \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\ &\quad = \int_{-\alpha}^{0} \int_{0}^{\sigma+\alpha} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_{0}^{T-\alpha} \int_{\sigma}^{\sigma+\alpha} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_{T-\alpha}^{T} \int_{\sigma}^{T} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma \\ &\quad = \int_{-\alpha}^{0} (\alpha + \sigma)\left|x'(\sigma)\right|^{n_1} d\sigma + \alpha \int_{0}^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma + \int_{T-\alpha}^{T} (T - \sigma)\left|x'(\sigma)\right|^{n_1} d\sigma.\end{aligned} \tag{2.3}$$

If we take the substitution of $u = \sigma - T$, then

$$\int_{T-\alpha}^{T} (T - \sigma)\left|x'(\sigma)\right|^{n_1} d\sigma = -\int_{-\alpha}^{0} u\left|x'(u + T)\right|^{n_1} du = -\int_{-\alpha}^{0} u\left|x'(u)\right|^{n_1} du. \tag{2.4}$$

By substituting (2.4) into (2.3), we obtain

$$\begin{aligned}
&\int_0^T \int_{t-\alpha}^t \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&\quad = \int_{-\alpha}^0 (\alpha+\sigma)\left|x'(\sigma)\right|^{n_1} d\sigma - \int_{-\alpha}^0 u\left|x'(u)\right|^{n_1} du + \alpha \int_0^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma \\
&\quad = \alpha \int_{-\alpha}^0 \left|x'(\sigma)\right|^{n_1} d\sigma + \alpha \int_0^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma = \alpha \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma \\
&\quad = \alpha \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma,
\end{aligned}$$

and then substituting the above formula into (2.2), we have

$$\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt \leqslant \alpha^{\frac{n_1}{m}} \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma.$$

Thus, formula (2.1) holds.

**Case 2.** $\alpha \in (T, +\infty)$. By interchanging the order of $\int_0^T \int_{t-\alpha}^t |x'(\sigma)|^{n_1} d\sigma\, dt$, we obtain

$$\begin{aligned}
&\int_0^T \int_{t-\alpha}^t \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&\quad = \int_{-\alpha}^{T-\alpha} \int_0^{\sigma+\alpha} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_{T-\alpha}^0 \int_0^T \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_0^T \int_\sigma^T \left|x'(\sigma)\right|^{n_1} dt\, d\sigma \\
&\quad = \int_{-\alpha}^{T-\alpha} (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + T \int_{T-\alpha}^0 \left|x'(\sigma)\right|^{n_1} d\sigma + \int_0^T (T-\sigma)\left|x'(\sigma)\right|^{n_1} d\sigma \\
&\quad = \int_{-\alpha}^0 (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + \int_0^T (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + \int_T^{T-\alpha} (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma \\
&\qquad + T \int_{T-\alpha}^0 \left|x'(\sigma)\right|^{n_1} d\sigma + \int_0^T (T-\sigma)\left|x'(\sigma)\right|^{n_1} d\sigma \\
&\quad = (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma + \int_{-\alpha}^0 (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + \int_T^{T-\alpha} (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma
\end{aligned}$$

$$+ T \int_{T-\alpha}^{0} \left|x'(\sigma)\right|^{n_1} d\sigma. \tag{2.5}$$

If we take $u = \sigma - T$, then $\int_T^{T-\alpha} (\sigma + \alpha)|x'(\sigma)|^{n_1} d\sigma = \int_0^{-\alpha} (u + T + \alpha)|x'(u)|^{n_1} du$. It follows from (2.5) that

$$\begin{aligned}
&\int_0^T \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma + \int_{-\alpha}^{0} (\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + T \int_{T-\alpha}^{0} \left|x'(\sigma)\right|^{n_1} d\sigma \\
&\quad + \int_0^{-\alpha} (\sigma + T + \alpha)\left|x'(\sigma)\right|^{n_1} d\sigma \\
&= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma - T \int_{-\alpha}^{0} \left|x'(\sigma)\right|^{n_1} d\sigma - T \int_0^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma \\
&= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma - T \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma \\
&= \alpha \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma.
\end{aligned}$$

By substituting the above formula into (2.2), we have

$$\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt \leqslant \alpha^{1+\frac{n_1}{m}} \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma = \alpha^{n_1} \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma.$$

So (2.1) holds.

2. If $s(t) \in [-\alpha, 0]$, $\forall t \in [0, T]$, then $\forall t \in [0, T]$. By using Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt &= \int_0^T \left| \int_t^{t-s(t)} x'(\sigma)\, d\sigma \right|^{n_1} dt \\
&\leqslant \int_0^T \left( \int_t^{t-s(t)} \left|x'(\sigma)\right| d\sigma \right)^{n_1} dt
\end{aligned}$$

$$\leqslant \left|s(t)\right|^{\frac{n_1}{m}} \int_0^T \int_t^{t-s(t)} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt$$

$$\leqslant \alpha^{\frac{n_1}{m}} \int_0^T \int_t^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt. \tag{2.6}$$

**Case 1.** If $\alpha \in [0, T]$, then by interchanging the order of $\int_0^T \int_t^{t+\alpha} |x'(\sigma)|^{n_1} d\sigma\, dt$, we have

$$\begin{aligned}
&\int_0^T \int_t^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&\quad = \int_0^\alpha \int_0^\sigma \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_\alpha^T \int_{\sigma-\alpha}^{\sigma} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_T^{T+\alpha} \int_{\sigma-\alpha}^{T} \left|x'(\sigma)\right|^{n_1} dt\, d\sigma \\
&\quad = \int_0^\alpha \sigma \left|x'(\sigma)\right|^{n_1} d\sigma + \alpha \int_\alpha^T \left|x'(\sigma)\right|^{n_1} d\sigma + \int_T^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma.
\end{aligned} \tag{2.7}$$

If we take $u = \sigma - T$, then

$$\begin{aligned}
\int_T^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma &= \int_0^\alpha (\alpha-u)\left|x'(u+T)\right|^{n_1} du \\
&= \int_0^\alpha (\alpha-u)\left|x'(u)\right|^{n_1} du.
\end{aligned}$$

It follows from (2.7) that

$$\int_0^T \int_t^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt = \alpha \int_0^\alpha \left|x'(\sigma)\right|^{n_1} d\sigma + \alpha \int_\alpha^T \left|x'(\sigma)\right|^{n_1} d\sigma = \alpha \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma.$$

Substituting the above formula into (2.6), we have

$$\int_0^T \left|x(t) - x\big(t-s(t)\big)\right|^{n_1} dt \leqslant \alpha^{n_1} \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma.$$

**Case 2.** If $\alpha \in (T, \infty)$, then by interchanging the order of $\int_0^T \int_t^{t+\alpha} |x'(\sigma)|^{n_1} d\sigma\, dt$, we have

$$\int_0^T \int_t^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt$$

$$= \int_0^T \int_0^\sigma \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_T^\alpha \int_0^T \left|x'(\sigma)\right|^{n_1} dt\, d\sigma + \int_\alpha^{T+\alpha} \int_{\sigma-\alpha}^T \left|x'(\sigma)\right|^{n_1} dt\, d\sigma$$

$$= \int_0^T \sigma \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_T^\alpha \left|x'(\sigma)\right|^{n_1} d\sigma + \int_\alpha^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma$$

$$= \int_0^T \sigma \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_T^\alpha \left|x'(\sigma)\right|^{n_1} d\sigma + \int_\alpha^0 (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma$$

$$+ \int_0^T (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma + \int_T^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma$$

$$= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_T^\alpha \left|x'(\sigma)\right|^{n_1} d\sigma + \int_\alpha^0 (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma$$

$$+ \int_T^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma. \tag{2.8}$$

If we take $u = \sigma - T$, then

$$\int_T^{T+\alpha} (T-\sigma+\alpha)\left|x'(\sigma)\right|^{n_1} d\sigma = \int_0^\alpha (\alpha-u)\left|x'(u+T)\right|^{n_1} du$$

$$= \int_0^\alpha (\alpha-u)\left|x'(u)\right|^{n_1} du.$$

So it follows from (2.8) that

$$\int_0^T \int_t^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt$$

$$= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_T^\alpha \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_\alpha^0 \left|x'(\sigma)\right|^{n_1} d\sigma$$

$$= (T+\alpha) \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma + T \int_T^0 \left|x'(\sigma)\right|^{n_1} d\sigma = \alpha \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma,$$

which together with (2.6) yields

$$\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt \leqslant \alpha^{n_1} \int_0^T \left|x'(\sigma)\right|^{n_1} d\sigma. \qquad \square$$

**Corollary 2.1.** *Let $n_1 > 1$, $\alpha \in [0, \infty)$ be two constants, $s \in C(R, R)$ with $s(t+T) = s(t)$, $\forall t \in [0, T]$ and $s(t) \in [-\alpha, \alpha]$, $\forall t \in [0, T]$. Then for $\forall x \in C^1(R, R)$ with $x(t+T) \equiv x(t)$, we have*

$$\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt \leqslant 2\alpha^{n_1} \int_0^T \left|x'(t)\right|^{n_1} dt. \tag{2.9}$$

**Proof.** Let $\Delta_1 = \{t\colon\ t \in [0, T],\ s(t) \geqslant 0\}$, $\Delta_2 = \{t\colon\ t \in [0, T],\ s(t) < 0\}$. Then for $\forall t \in [0, T]$,

$$\begin{aligned}
\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt &= \int_{\Delta_1 \cup \Delta_2} \left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} dt \\
&= \int_{\Delta_1} \left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} dt + \int_{\Delta_2} \left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} dt \\
&= \int_{\Delta_1} \left| \int_{t-s(t)}^{t} x'(\sigma)\, d\sigma \right|^{n_1} dt + \int_{\Delta_2} \left| \int_{t}^{t-s(t)} x'(\sigma)\, d\sigma \right|^{n_1} dt.
\end{aligned}$$

By using Hölder's inequality, we get

$$\begin{aligned}
&\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^{n_1} dt \\
&\quad \leqslant \left|s(t)\right| \int_{\Delta_1} \int_{t-s(t)}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt + \left|s(t)\right| \int_{\Delta_2} \int_{t}^{t-s(t)} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&\quad \leqslant \alpha \int_{\Delta_1} \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt + \alpha \int_{\Delta_2} \int_{t}^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt \\
&\quad \leqslant \alpha \int_0^T \int_{t-\alpha}^{t} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt + \alpha \int_0^T \int_{t}^{t+\alpha} \left|x'(\sigma)\right|^{n_1} d\sigma\, dt.
\end{aligned}$$

Thus, the result (2.9) follows from Lemma 2.2 immediately. $\square$

**Remark 2.1.** If $n_1 = 2$ and $\alpha \in (0, T]$, then (2.9) was proved by [5]. So Corollary 2.1 generalizes the corresponding work of [5].

First, we recall Mawhin's continuation theorem which our study is based upon.

Let $X$ and $Y$ be real Banach spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of $L$. This means that $\operatorname{Im} L$ is closed in $Y$ and $\dim \ker L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces $X_1$ and $Y_1$ such that $X = \ker L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$ and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by $K$ the inverse of $L_P$.

Now, let $\Omega$ be an open bounded subset of $X$ with $D(L) \cap \Omega \neq \emptyset$. A map $N : \bar{\Omega} \to Y$ is said to be $L$-compact in $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and the operator $K(I - Q)N : \bar{\Omega} \to X$ is compact.

**Lemma 2.2** [8]. *Suppose that $X$ and $Y$ are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is $L$-compact on $\bar{\Omega}$. If*

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \ker L$; *and*
(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \ker L$ *is an isomorphism,*

*then the equation $Lx = Nx$ has a solution in $\bar{\Omega} \cap D(L)$.*

In order to use Mawhin's continuation theorem, we set

$$Y = C_T = \{x\colon\ x \in C(R, R),\ x(t + T) \equiv x(t)\}$$

with the norm $|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|$, $\forall \varphi \in C_T$, and

$$X = C_T^1 = \{x\colon\ x \in C^1(R, R),\ x(t + T) \equiv x(t)\}$$

with the norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}$. Clearly, $X$ and $Y$ are two Banach spaces. We also define the operators $L$ and $N$ as follows:

$$L : D(L) \subset X \to Y, \quad Lx = x'', \tag{2.10}$$

where $D(L) = \{x\colon\ x \in C^2(R, R),\quad x(t + T) \equiv x(t)\}$;

$$N : X \to Y, \quad [Nx](t) = f\big(t, x(t),\ x\big(t - \tau(t)\big), x'(t)\big) + e(t). \tag{2.11}$$

It is easy to see that Eq. (1.7) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of $L$, we see $\ker L = R$ and $\operatorname{Im} L = \{x\colon\ x \in Y,\ \int_0^T x(s)\,ds = 0\}$. Thus, $L$ is a Fredholm operator with index zero. Let

$$P : X \to \ker L, \qquad Q : Y \to Y/\operatorname{Im} L$$

be defined respectively by

$$Px = x(0), \qquad Qx = \frac{1}{T}\int_0^T x(s)\,ds$$

and let

$$L_P = L|_{X \cap \ker P} : X \cap \ker P \to \operatorname{Im} L.$$

Then $L_P$ has a unique continuous inverse $L_P^{-1}$ on $\operatorname{Im} L$ defined by

$$\left(L_P^{-1} y\right)(t) = \int_0^T G(t,s) y(s)\, ds, \tag{2.12}$$

where

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T}, & 0 \leqslant s < t \leqslant T, \\ \frac{t(s-T)}{T}, & 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

From (2.11) and (2.12), one can easily find that $N$ is $L$-compact in $\bar{\Omega}$, where $\Omega$ is an open bounded subset of $X$.

## 3. Main result

**Theorem 3.1.** *Suppose that the following conditions hold*:

$[H_1]$ *there is a constant* $d > 0$ *such that*

$$f(t, x_0, x_1, 0) > |e|_0, \quad \forall (t, x_0, x_1) \in [0,T] \times R^2 \text{ with } x_0 \geqslant x_1 > d;$$

$[H_1]$ *there is a constant* $d > 0$ *such that*

$$f(t, x_0, x_1, 0) < -|e|_0, \quad \forall (t, x_0, x_1) \in [0,T] \times R^2 \text{ with } x_0 \leqslant x_1 < -d;$$

$[H_3]$ *the function* $f$ *has the decomposition*

$$f(t, x_0, x_1, x_2) = u(t, x_0, x_1, x_2) + h(t, x_0) + g(t, x_1) + p(t, x_2) \tag{3.1}$$

*such that*

$$x_2 u(t, x_0, x_1, x_2) \leqslant -\beta |x_2|^{n+1}\, dt, \quad \forall (t, x_0, x_1, x_2) \in [0,T] \times R^3, \tag{3.2}$$

$$\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|h(t,x)|}{|x|^n} = r_0, \tag{3.3}$$

$$\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|g(t,x)|}{|x|^n} = r_1, \quad \text{and} \tag{3.4}$$

$$\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|p(t,x)|}{|x|^n} = r_2, \tag{3.5}$$

*where* $n \geqslant 1$, $\beta > 0$, $r_i \geqslant 0$, $i = 0, 1, 2$ *are all constants*, $g(t,x)$, $h(t,x)$, *and* $p(t,x)$ *are continuous on* $R \times R$ *with* $g(t+T, x) \equiv g(t,x)$, $h(t+T, x) \equiv h(t,x)$, *and* $p(t+T, x) \equiv p(t,x)$, $\forall x \in R$.

*Then Eq.* (1.7) *has at least one* $T$*-periodic solution, if* $(r_0 + r_1)T^n + r_2 < \beta$.

**Proof.** Let us consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

where $L$ and $N$ are defined by (2.10) and (2.11), respectively. Suppose $x \in D(L)$ is an arbitrary solution of equation $Lx = \lambda Nx$, for some $\lambda \in (0, 1)$. Then

$$x''(t) = \lambda f\bigl(t, x(t), x\bigl(t - \tau(t)\bigr), x'(t)\bigr) + \lambda e(t), \quad \lambda \in (0, 1). \tag{3.6}$$

Let $t_0$ be the global maximum point of $x(t)$ on $[0, T]$. Then $x'(t_0) = 0$, and $x''(t_0) \leqslant 0$. It follows from (3.6) that

$$\lambda f\bigl(t_0, x(t_0), x\bigl(t - \tau(t_0)\bigr), 0\bigr) + \lambda e(t_0) = x''(t_0) \leqslant 0,$$

i.e.,

$$f\bigl(t_0, x(t_0), x\bigl(t - \tau(t_0)\bigr), 0\bigr) \leqslant -e(t_0) \leqslant |e|_0.$$

In view of $x(t_0) \geqslant x(t_0 - \tau(t_0))$, we have from assumption $[H_1]$ that

$$x\bigl(t_0 - \tau(t_0)\bigr) \leqslant d. \tag{3.7}$$

In the same way, if we set $t_1$ to represent the minimum point of $x(t)$ on $[0, T]$, then by using assumption $[H_2]$, we can obtain

$$x\bigl(t_1 - \tau(t_1)\bigr) \geqslant -d. \tag{3.8}$$

Now, we begin to prove that there is a constant $\xi \in R$ such that

$$\bigl|x(\xi)\bigr| \leqslant d. \tag{3.9}$$

(1) If $x(t_1 - \tau(t_1)) > d$, then from (3.7) and the continuity of $x(t)$ on $[0, T]$, we see that there is a constant $t_2 \in [0, T]$ such that

$$x\bigl(t_2 - \tau(t_2)\bigr) = d. \tag{3.10}$$

(2) If $x(t_1 - \tau(t_1)) \leqslant d$, then from (3.8) we have

$$\bigl|x\bigl(t_1 - \tau(t_1)\bigr)\bigr| \leqslant d. \tag{3.11}$$

From (3.10) and (3.11), we see in either case (1) or case (2) that (3.9) always holds. Since $\xi \in R$ is a constant, there must be an integer $k$ and a point $t_3 \in [0, T]$ such that $\xi = kT + t_3$. So $|x(t_3)| = |x(\xi)| \leqslant d$, which implies

$$|x|_0 \leqslant d + \int_0^T \bigl|x'(s)\bigr|\,ds. \tag{3.12}$$

On the other hand, multiplying the two sides of (3.6) by $x'(t)$ and integrating them over the interval $[0, T]$, we have

$$\int_0^T x'(t) f\bigl(t, x(t), x\bigl(t - \tau(t)\bigr), x'(t)\bigr)\,dt + \int_0^T x'(t)e(t) = 0.$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
\beta\int_0^T \left|x'(t)\right|^{n+1} dt &\leqslant -\int_0^T x'(t)u\big(t,x(t),x\big(t-\tau(t)\big),x'(t)\big)\\
&\leqslant \int_0^T \left|x'(t)h\big(t,x(t)\big)\right| dt+\int_0^T \left|x'(t)g\big(t,x\big(t-\tau(t)\big)\big)\right| dt\\
&\quad+\int_0^T \left|x'(t)p\big(t,x'(t)\big)\right| dt \int_0^T \left|x'(t)e(t)\right| dt. \qquad (3.13)
\end{aligned}$$

Let

$$\varepsilon=\frac{\beta-(r_0+r_1)T^n-r_2}{2(2T^n+1)}.$$

From the assumption $(r_0+r_1)T^n+r_2<\beta$, we see $\varepsilon>0$. For such a $\varepsilon>0$, one can find from assumptions (3.2)–(3.5) that there is a constant $D>0$ such that

$$\begin{aligned}
&\frac{|h(t,x)|}{|x|^n}<(r_0+\varepsilon),\quad \text{uniformly for } t\in[0,T],\ |x|>D,\\
&\frac{|g(t,x)|}{|x|^n}<(r_1+\varepsilon),\quad \text{uniformly for } t\in[0,T],\ |x|>D,\quad \text{and}\\
&\frac{|p(t,x)|}{|x|^n}<(r_2+\varepsilon),\quad \text{uniformly for } t\in[0,T],\ |x|>D,
\end{aligned}$$

i.e.,

$$\left|h(t,x)\right|<(r_0+\varepsilon)|x|^n,\quad \text{uniformly for } t\in[0,T],\ |x|>D, \qquad (3.14)$$

$$\left|g(t,x)\right|<(r_1+\varepsilon)|x|^n,\quad \text{uniformly for } t\in[0,T],\ |x|>D,\quad \text{and} \qquad (3.15)$$

$$\left|p(t,x)\right|<(r_0+\varepsilon)|x|^n,\quad \text{uniformly for } t\in[0,T],\ |x|>D. \qquad (3.16)$$

Let

$$\begin{aligned}
&\Delta_1=\left\{t:\ t\in[0,T],\ \left|x(t)\right|\leqslant D\right\},\\
&\Delta_2=\left\{t:\ t\in[0,T],\ \left|x(t)\right|>D\right\},\\
&\Delta_3=\left\{t:\ t\in[0,T],\ \left|x\big(t-\tau(t)\big)\right|\leqslant D\right\},\\
&\Delta_4=\left\{t:\ t\in[0,T],\ \left|x\big(t-\tau(t)\big)\right|>D\right\},\\
&\Delta_5=\left\{t:\ t\in[0,T],\ \left|x'(t)\right|\leqslant D\right\},\quad \text{and}\\
&\Delta_6=\left\{t:\ t\in[0,T],\ \left|x'(t)\right|>D\right\}.
\end{aligned}$$

Then we have from (3.13) that

$$\begin{aligned}
\beta \int_0^T \left|x'(t)\right|^{n+1} dt
&\leqslant \int_{\Delta_1} \left|h\big(s, x(s)\big)x'(s)\right| ds + \int_{\Delta_2} \left|h\big(s, x(s)\big)x'(s)\right| ds \\
&\quad + \int_{\Delta_3} \left|g\big(s, x\big(s-\tau(s)\big)\big)x'(s)\right| ds + \int_{\Delta_4} \left|g\big(s, x\big(s-\tau(s)\big)\big)x'(s)\right| ds \\
&\quad + \int_{\Delta_5} \left|p\big(s, x'(s)\big)x'(s)\right| ds + \int_{\Delta_6} \left|p\big(s, x'(s)\big)x'(s)\right| ds + \int_0^T \left|e(t)x'(t)\right| dt. \qquad (3.17)
\end{aligned}$$

It follows from (3.14)–(3.17) that

$$\begin{aligned}
\beta \int_0^T \left|x'(t)\right|^{n+1} dt &\leqslant (r_0 + r_1 + 2\varepsilon)|x|_0^n \int_0^T \left|x'(s)\right| ds + (r_2 + \varepsilon) \int_0^T \left|x'(s)\right|^{n+1} ds \\
&\quad + \left[g_D + h_D + p_D + |e|_0\right] \int_0^T \left|x'(s)\right| ds, \qquad (3.18)
\end{aligned}$$

where

$$g_D = \max_{t\in[0,T],\,|x|\leqslant D} \left|g(t,x)\right|, \qquad h_D = \max_{t\in[0,T],\,|x|\leqslant D} \left|h(t,x)\right|, \quad \text{and}$$
$$p_D = \max_{t\in[0,T],\,|x|\leqslant D} \left|p(t,x)\right|.$$

By classical elementary inequalities, we see that for each $k \geqslant 1$, there is a $\delta(k) > 0$ which dependent on $k$ only, such that

$$(1+x)^k < 1 + (k+1)x, \quad x \in \big(0, \delta(k)\big]. \qquad (3.19)$$

So

$$(1+x)^n < 1 + (n+1)x, \quad x \in \big(0, \delta(n)\big). \qquad (3.20)$$

From condition $f(t,c,c,0) + e(t) \not\equiv 0$, where $c \in R$ is an abstract constant, it is easy to see that $x'(t) \not\equiv 0$, i.e., $\int_0^T |x_1'(s)|\,ds > 0$.

We will prove next that there must be a constant $M_1 > 0$ independent of $\lambda$ such that

$$\int_0^T \left|x'(s)\right|^{n+1} ds \leqslant M_1. \qquad (3.21)$$

**Case (1).** If $d/\int_0^T |x'(s)|\,ds > \delta(n)$, then $\int_0^T |x'(s)|\,ds < d/\delta(n)$, i.e.,

$$|x|_0 \leqslant d + d/\delta(n) := G.$$

Substituting the above formula into (3.18), we have

$$\beta\int_0^T\left|x'(t)\right|^{n+1}dt\leqslant(r_2+\varepsilon)\int_0^T\left|x'(s)\right|^{n+1}ds+\big[(r_0+r_1+2\varepsilon)G^n+\big(g_D+h_D+p_D+|e|_0\big)\big]T^{\frac{n}{n+1}}\left(\int_0^T\left|x'(s)\right|^{n+1}ds\right)^{\frac{1}{n+1}}. \tag{3.22}$$

Since $\beta>r_2+\varepsilon$ and $\frac{1}{n+1}<1$, it follows from (3.22) that there is a constant $G_1>0$ such that

$$\int_0^T\left|x'(s)\right|^{n+1}ds\leqslant G_1. \tag{3.23}$$

**Case (2).** If $d/\int_0^T|x'(s)|\,ds\leqslant\delta(n)$, it follows from (3.12) and (3.20) that

$$\begin{aligned}|x|_0^n&\leqslant\left(d+\int_0^T\left|x'(s)\right|ds\right)^n=\left(\int_0^T\left|x'(t)\right|dt\right)^n\left(1+\frac{d}{\int_0^T|x'(t)|\,dt}\right)^n\\&\leqslant\left(\int_0^T\left|x'(t)\right|dt\right)^n\left[1+\frac{(n+1)d}{\int_0^T|x'(t)|\,dt}\right]\\&=\left(\int_0^T\left|x'(t)\right|dt\right)^n+(n+1)d\left(\int_0^T\left|x'(t)\right|dt\right)^{n-1},\end{aligned}$$

which together with Hölder's inequality results in

$$\begin{aligned}|x|_0^n\int_0^T\left|x'(s)\right|ds&\leqslant\left(\int_0^T\left|x'(t)\right|dt\right)^{n+1}+(n+1)d\left(\int_0^T\left|x'(t)\right|dt\right)^n\\&\leqslant\left(T^{\frac{n}{n+1}}\left(\int_0^T\left|x'(t)\right|^{n+1}dt\right)^{\frac{1}{n+1}}\right)^{n+1}\\&\quad+(n+1)d\left(T^{\frac{n}{n+1}}\left(\int_0^T\left|x'(t)\right|^{n+1}dt\right)^{\frac{1}{n+1}}\right)^n\\&=T^n\int_0^T\left|x'(t)\right|^{n+1}dt+(n+1)\,dT^{\frac{n^2}{n+1}}\left(\int_0^T\left|x'(t)\right|^{n+1}dt\right)^{\frac{n}{n+1}}.\end{aligned}$$

It follows from (3.18) that

$$\beta \int_0^1 \left|x'(t)\right|^{n+1} dt \leqslant \left[(r_0 + r_1 + 2\varepsilon)T^n + r_2 + \varepsilon\right] \int_0^T \left|x'(t)\right|^{n+1} dt$$
$$+ (n+1)dT^{\frac{n^2}{n+1}}(r_0 + r_1 + 2\varepsilon)\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{n}{n+1}}$$
$$+ \left[g_D + h_D + p_D + |e|_0\right]T^{\frac{n}{n+1}}\left(\int_0^T \left|x'(s)\right|^{n+1} ds\right)^{\frac{1}{n+1}}. \quad (3.24)$$

In view of

$$\varepsilon = \frac{\beta - (r_0 + r_1)T^n - r_2}{2(2T^n + 1)},$$

we see

$$(r_0 + r_1 + 2\varepsilon)T^n + r_2 + \varepsilon = \frac{\beta + (r_0 + r_1)T^n + r_2}{2} < \beta,$$

and $\frac{n}{n+1} < 1$, $\frac{1}{n+1} < 1$, it follows from (3.24) that there is a constant $G_2 > 0$ such that

$$\int_0^T \left|x'(s)\right|^{n+1} ds \leqslant G_2. \quad (3.25)$$

Let $M_1 = \max\{G_1, G_2\}$. Then from (3.23) and (3.25), we see in either case (1) or case (2), we have $\int_0^T |x'(s)|^{n+1} ds \leqslant M_1$, i.e., (3.21) holds, which yields from (3.12) that

$$|x|_0 \leqslant d + \int_0^T \left|x'(s)\right| ds \leqslant d + T^{\frac{n}{n+1}} M_1^{\frac{1}{n+1}} := M_0. \quad (3.26)$$

Since $x(0) = x(T)$, there must be a constant $\eta \in [0, T]$ such that $x'(\eta) = 0$; and then multiplying the two sides of Eq. (3.4) by $x'(t)$ and integrating them over $[\eta, t]$, where $t \in [\eta, \eta + T]$, we get

$$\frac{1}{2}\left|x'(t)\right|^2 = \lambda \int_\eta^t u\big(s, x(s), x\big(s - \tau(s), x'(s)\big)\big)x'(s)\, ds + \lambda \int_\eta^t h\big(s, x(s)\big)x'(s)\, ds$$
$$+ \lambda \int_\eta^t g\big(s, x\big(s - \tau(s)\big)\big)x'(s)\, ds + \lambda \int_\eta^t p\big(s, x'(s)\big)x'(s)\, ds$$
$$+ \lambda \int_\eta^t e(s)x'(s)\, ds$$

$$
\begin{aligned}
&\leqslant \int_{\eta}^{\eta+T} \left|h\big(s,x(s)\big)x'(s)\right| ds + \int_{\eta}^{\eta+T} \left|g\big(s,x\big(s-\tau(s)\big)\big)x'(s)\right| ds \\
&\quad + \int_{\eta}^{\eta+T} \left|p\big(s,x'(s)\big)x'(s)\right| ds + \int_{\eta}^{\eta+T} \left|e(s)x'(s)\right| ds \\
&= \int_0^T \left|h\big(s,x(s)\big)x'(s)\right| ds + \int_0^T \left|g\big(s,x\big(s-\tau(s)\big)\big)x'(s)\right| ds \\
&\quad + \int_0^T \left|p\big(s,x'(s)\big)x'(s)\right| ds + \int_0^T \left|e(s)x'(s)\right| ds \\
&= \int_0^T \left|h\big(s,x(s)\big)x'(s)\right| ds + \int_0^T \left|g\big(s,x\big(s-\tau(s)\big)\big)x'(s)\right| ds \\
&\quad + \int_{\Delta_5} \left|p\big(s,x'(s)\big)x'(s)\right| ds + \int_{\Delta_6} \left|p\big(s,x'(s)\big)x'(s)\right| ds + \int_0^T \left|e(s)x'(s)\right| ds \\
&\leqslant \big[g_{M_0} + h_{M_0} + p_D + |e|_0\big] \int_0^T \left|x'(s)\right| ds + (r_2+\varepsilon) \int_0^T \left|x'(t)\right|^{n+1} dt \\
&\leqslant \big[g_{M_0} + h_{M_0} + p_D + |e|_0\big] T^{\frac{n}{n+1}} \left( \int_0^T \left|x'(s)\right|^{n+1} ds \right)^{\frac{1}{n+1}} \\
&\quad + (r_2+\varepsilon) \int_0^T \left|x'(t)\right|^{n+1} dt,
\end{aligned}
$$

where $g_{M_0} = \max_{t\in[0,T],\,|x|\leqslant M_0} |g(t,x)|$ and $h_{M_0} = \max_{t\in[0,T],\,|x|\leqslant M_0} |h(t,x)|$. Substituting (3.21) into the above formula, we get

$$
\begin{aligned}
&\left|x'(t)\right|^2 \leqslant 2\Big[\big(h_{M_0} + g_{M_0} + p_D + |e|_0\big) T^{\frac{n}{n+1}} M_1^{\frac{1}{n+1}} + (r_2+\varepsilon) M_1\Big] := M_2, \\
&\quad \forall t \in [\eta, \eta+T],
\end{aligned}
$$

that is

$$
|x'|_0 = \max_{t\in[0,T]} \left|x'(t)\right| = \max_{t\in[\eta,\eta+T]} \left|x'(t)\right| \leqslant M_2,
$$

i.e.,

$$
\|x\| = \max\big\{|x|_0, |x'|_0\big\} \leqslant \max\{M_0, M_2\}. \tag{3.27}
$$

Suppose $x \in \ker L$ and $Nx \in \operatorname{Im} L$. Then $x = c \in R$ is a constant with $\int_0^T [f(s,c,c,0) + e(s)]\,ds = 0$. So we have from assumptions $[H_1]$ and $[H_2]$ that

$$|c| \leqslant d. \tag{3.28}$$

Now, if we set $\Omega = \{x \in X\colon |x|_0 < M_0 + 1,\ |x'|_0 < M_2 + 1\}$, then we easily see from (3.27), (3.28) that conditions (1) and (2) of Lemma 2.2 are satisfied. In what follows, we will show that condition (3) of Lemma 2.2 is also satisfied. In order to do it, we take

$$\begin{aligned} &H(x,\mu) : \bar{\Omega} \times [0,1] \\ &\quad \to X : H(x,\mu) = \mu x + \frac{1-\mu}{T} \int_0^T \bigl[f\bigl(t, x(t), x\bigl(t-\tau(t)\bigr), x'(t)\bigr) + e(t)\bigr]\,dt. \end{aligned}$$

From assumptions $[H_1]$ and $[H_2]$, we can easily obtain

$$H(x,\mu) \neq 0, \quad \forall (x,\mu) \in \Omega \cap \ker L \times [0,1],$$

which results in

$$\begin{aligned} \deg\{JQN, \Omega \cap \ker L, 0\} &= \deg\bigl\{H(x,0), \Omega \cap \ker L, 0\bigr\} \\ &= \deg\bigl\{H(x,1), \Omega \cap \ker L, 0\bigr\} \neq 0, \end{aligned}$$

where $J : R \to R : J(x) = x$. Hence, by using Lemma 2.2, we know that Eq. (1.7) has at least one $T$-periodic solution. □

**Theorem 3.2.** *Suppose that conditions $[H_1]$, $[H_2]$ of Theorem 3.1 hold. Furthermore, we assume that the function $f$ has the decomposition*

$$f(t,x_0,x_1,x_2) = u(t,x_0,x_1,x_2) + h(x_0) + g(x_1) + p(t,x_2) \tag{3.29}$$

*such that*

$$x_2 u(t,x_0,x_1,x_2) \leqslant -\beta |x_2|^{n+1}\,dt, \quad \forall (t,x_0,x_1,x_2) \in [0,T] \times R^3, \tag{3.30}$$

$$\bigl|g(x) - g(y)\bigr| \leqslant s_l |x-y|^l + \sum_{k=1}^{l-1} a_k(y)|x-y|^k \tag{3.31}$$

*and*

$$\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|p(t,x)|}{|x|^n} = r, \tag{3.32}$$

*where $a_k(y) \in C(R, R^+)$ satisfies*

$$\lim_{|y| \to +\infty} \frac{a_k(y)}{|x|^{l-k}} = s_k \quad (k = 1,2,\ldots,l-1), \tag{3.33}$$

*where $1 \leqslant l \leqslant n$ is a positive integer, $\beta > 0$, $n \geqslant 1$, $r \geqslant 0$, $s_i \geqslant 0$, $i = 0,1,\ldots,l-1$ are all constants, $g(x)$ is continuous on $R$ and $p(t,x)$ is continuous on $R \times R$ with $p(t+T,x) \equiv p(t,x)$, $\forall x \in R$. Then Eq.* (1.7) *has at least one $T$-periodic solution, if one of the following conditions holds*:

$[A_1]$ *$l = n$ and*

$$r + 2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k} < \beta,$$

$[A_2]$ *$l < n$ and $r < \beta$.*

**Proof.** Since assumptions $[H_1]$, $[H_2]$ hold, from the proof of Theorem 3.1 we see that there is a point $\xi \in [0, T]$ such that $|x(\xi)| \leqslant d$, i.e.,

$$|x|_0 \leqslant d + \int_0^T \left|x'(s)\right| ds. \tag{3.34}$$

Multiplying the two sides of (3.4) by $x'(t)$, and integrating them on $[0, T]$, we have

$$\begin{aligned}
&\int_0^T u\big(t, x(t), x\big(t - \tau(t)\big), x'(t)\big)x'(t)\,dt \\
&\quad = -\int_0^T h\big(x(t)\big)x'(t)\,dt - \int_0^T g\big(x\big(t - \tau(t)\big)\big)x'(t)\,dt \\
&\qquad - \int_0^T p\big(t, x'(t)\big)x'(t)\,dt - \int_0^T e(t)x'(t)\,dt.
\end{aligned}$$

By (3.30), $\int_0^T g(x(t))x'(t)\,dt = 0$ and $\int_0^T h(x(t))x'(t)\,dt = 0$; we see from the above formula that

$$\begin{aligned}
&\beta \int_0^T \left|x'(t)\right|^{n+1} dt \\
&\quad \leqslant \left|\int_0^T g\big(x\big(t - \tau(t)\big)\big)x'(t)\,dt\right| + \left|\int_0^T p\big(t, x'(t)\big)x'(t)\,dt\right| + \left|\int_0^T e(t)x'(t)\,dt\right| \\
&\quad = \left|\int_0^T \big[g\big(x(t)\big) - g\big(x\big(t - \tau(t)\big)\big)\big]x'(t)\,dt\right| \\
&\qquad + \left|\int_0^T p\big(t, x'(t)\big)x'(t)\,dt\right| + \left|\int_0^T e(t)x'(t)\,dt\right|.
\end{aligned} \tag{3.35}$$

Let

$$\varepsilon = \begin{cases} \frac{\beta - r - 2^{\frac{l}{n+1}} r_l |\tau|_0^l - \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} r_k |\tau|_0^k T^{n-k}}{2(1+\sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} |\tau|_0^k T^{n-k})}, & \text{if } [A_1] \text{ is satisfied;} \\ \frac{\beta - r}{2}, & \text{if } [A_2] \text{ is satisfied.} \end{cases}$$

Clearly, $\varepsilon > 0$ is a constant independent of $\lambda$, and if condition $[A_1]$ holds,

$$\begin{aligned} & r + \varepsilon + 2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} (s_k + \varepsilon) |\tau|_0^k T^{n-k} \\ &\quad = r + \varepsilon + 2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k} + \varepsilon \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} |\tau|_0^k T^{n-k} \\ &\quad \leqslant r + 2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k} \\ &\qquad + \frac{1}{2}\left(\beta - r - 2^{\frac{l}{n+1}} s_l |\tau|_0^l - \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k}\right) \\ &\quad = \frac{1}{2}\left(\beta + r + 2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k}\right) < \beta, \end{aligned} \tag{3.36}$$

and also, if condition $[A_2]$ holds, then

$$r + \varepsilon < \beta. \tag{3.37}$$

For the above constant $\varepsilon > 0$, from (3.32) and (3.33) we see that there is a constant $\rho > d$ such that

$$\left|p(t,x)\right| < (r+\varepsilon)|x|^n, \quad \text{uniformly for } t \in [0,T],\ |x| > \rho \tag{3.38}$$

and

$$a_k(y) < (r_k + \varepsilon)|y|^{l-k}, \quad \text{for } |x| > \rho\ (k = 1, 2, \ldots, l-1). \tag{3.39}$$

Let

$$\begin{aligned} & E_1 = \left\{t\colon\ t \in [0,T],\ \left|x\left(t - \tau(t)\right)\right| \leqslant \rho\right\}, \\ & E_2 = \left\{t\colon\ t \in [0,T],\ \left|x\left(t - \tau(t)\right)\right| > \rho\right\}, \\ & E_3 = \left\{t\colon\ t \in [0,T],\ \left|x'(t)\right| > \rho\right\}, \quad \text{and} \quad E_4 = \left\{t\colon\ t \in [0,T],\ \left|x'(t)\right| \leqslant \rho\right\}, \end{aligned}$$

and then it follows from (3.38) and (3.32) that

$$\begin{aligned} \left|\int_0^T p\left(t, x'(t)\right) x'(t)\, dt\right| &\leqslant \int_0^T \left|p\left(t, x'(t)\right) x'(t)\right| dt \\ &\leqslant \int_{E_3} \left|p\left(t, x'(t)\right) x'(t)\right| dt + \int_{E_4} \left|p\left(t, x'(t)\right) x'(t)\right| dt \end{aligned}$$

$$\leqslant p_\rho \int_0^T |x'(t)|\,dt + (r+\varepsilon)\int_0^T |x'(t)|^{n+1}\,dt, \tag{3.40}$$

where $p_\rho = \max_{t\in[0,T],\,|x|\leqslant\rho} |p(t,x)|$. Furthermore, from (3.31), (3.33), and (3.39), we get

$$\begin{aligned}
&\left|\int_0^T x'(t)\big[g\big(x(t)\big) - g\big(x\big(t-\tau(t)\big)\big)\big]\,dt\right| \\
&\quad\leqslant s_l \int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^l\,dt \\
&\qquad + \sum_{k=1}^{l-1}\int_0^T |x'(t)|\big|a_k\big(x\big(t-\tau(t)\big)\big)\big|\big|x(t) - x_1\big(t-\tau(t)\big)\big|^k\,dt \\
&\quad= s_l \int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^l\,dt \\
&\qquad + \sum_{k=1}^{l-1}\int_{E_1} |x'(t)|\big|a_k\big(x\big(t-\tau(t)\big)\big)\big|\big|x(t) - x\big(t-\tau(t)\big)\big|^k\,dt \\
&\qquad + \sum_{k=1}^{l-1}\int_{E_2} |x'(t)|\big|a_k\big(x\big(t-\tau(t)\big)\big)\big|\big|x(t) - x\big(t-\tau(t)\big)\big|^k\,dt \\
&\quad\leqslant s_l \int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^l\,dt \\
&\qquad + \sum_{k=1}^{l-1} a_{k,\rho}\int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^k\,dt \\
&\qquad + \sum_{k=1}^{l-1} (s_k+\varepsilon)|x|_0^{l-k}\int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^k\,dt,
\end{aligned} \tag{3.41}$$

where $a_{k,\rho} = \max_{|x|\leqslant\rho} a_k(x)$ $(k = 1, 2, \ldots, l-1)$. By using Hölder's inequality and Corollary 2.1, we obtain

$$\int_0^T |x'(t)|\big|x(t) - x\big(t-\tau(t)\big)\big|^l\,dt$$

$$\leqslant \left(\int_0^T \left|x(t)-x\big(t-\tau(t)\big)\right|^{n+1} dt\right)^{\frac{l}{n+1}} \left(\int_0^T \left|x'(t)\right|^{\frac{n+1}{n+1-l}} dt\right)^{\frac{n+1-l}{n+1}}$$

$$\leqslant 2^{\frac{l}{n+1}} |\tau|_0^l \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l}{n+1}} \left[T^{\frac{n-l}{n+1-l}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1-l}}\right]^{\frac{n+1-l}{n+1}}$$

$$= 2^{\frac{l}{n+1}} |\tau|_0^l T^{\frac{n-l}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l+1}{n+1}}. \tag{3.42}$$

Similarly, for $k = 1, 2, \ldots, l-1$ we have

$$\int_0^T \left|x'(t)\right|\left|x(t)-x\big(t-\tau(t)\big)\right|^k dt \leqslant 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}}. \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.41), we have

$$\begin{aligned}
&\left|\int_0^T x'(t)\big[g\big(x(t)\big)-g\big(x\big(t-\tau(t)\big)\big)\big] dt\right| \\
&\quad \leqslant 2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l+1}{n+1}} \\
&\qquad + \sum_{k=1}^{l-1} a_{k,\rho} 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}} \\
&\qquad + \sum_{k=1}^{l-1} (s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} |x|_0^{l-k} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}}.
\end{aligned}$$

Substituting the above formula, (3.12), and (3.40) into (3.35), we obtain

$$\begin{aligned}
\beta \int_0^T \left|x'(t)\right|^{n+1} dt \leqslant{}& 2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l+1}{n+1}} \\
&+ \sum_{k=1}^{l-1} a_{k,\rho} 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}} \\
&+ \sum_{k=1}^{l-1} (s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(d + \int_0^T \left|x'(s)\right| ds\right)^{l-k}
\end{aligned}$$

$$\times\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}} + (r+\varepsilon)\int_0^T \left|x'(t)\right|^{n+1} dt$$
$$+\left[p_\rho + \left(\int_0^T \left|e(t)\right|^{\frac{n+1}{n}} dt\right)^{\frac{n}{n+1}}\right]\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}}. \tag{3.44}$$

From (3.19), we see there is a constant $\delta(l) > 0$ which is dependent on $l$ only, such that

$$(1+x)^{l-k} < (1+x)^l < 1+(l+1)x, \quad x \in \big(0, \delta(l)\big],\ k = 1, 2, \ldots, l-1. \tag{3.45}$$

From condition $f(t, c, c, 0) + e(t) \not\equiv 0$, where $c \in R$ is an arbitrary constant, it is easy to see that $x'(t) \not\equiv 0$. So $\int_0^T |x'(s)|\,ds > 0$.

We will prove next that if one of conditions of $[A_1]$ and $[A_2]$ holds, then there must be a constant $M > 0$ independent of $\lambda$ such that

$$\int_0^T \left|x'(s)\right|^{n+1} ds \leqslant M. \tag{3.46}$$

**Case 1.** If $d/\int_0^T |x'(s)|\,ds > \delta(l)$, then $\int_0^T |x'(s)|\,ds < d/\delta(l)$, it follows from (3.44) that

$$\beta\int_0^T \left|x'(t)\right|^{n+1} dt \leqslant 2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}}\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l+1}{n+1}}$$
$$+\sum_{k=1}^{l-1} a_{k,\rho} 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}}\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}}$$
$$+\sum_{k=1}^{l-1} (s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}}\big(d + d/\delta(l)\big)^{l-k}$$
$$\times\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}} + (r+\varepsilon)\int_0^T \left|x'(t)\right|^{n+1} dt$$
$$+\left[p_\rho + \left(\int_0^T \left|e(t)\right|^{\frac{n+1}{n}} dt\right)^{\frac{n}{n+1}}\right]\left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}}. \tag{3.47}$$

(1) If condition $[A_1]$ holds, then from (3.36), we see

$$\beta > 2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}} + r + \varepsilon.$$

Moreover, $\frac{k+1}{n+1} < 1$, for $k = 1, 2, \ldots, l-1$, and $\frac{1}{n+1} < 1$. So we have from (3.47) that there is a constant $M_1 > 0$ such that

$$\int_0^T \left|x'(t)\right|^{n+1} dt \leqslant M_1. \tag{3.48}$$

(2) If condition $[A_2]$ holds, then $\frac{l+1}{n+1} < 1$, $\frac{k+1}{n+1} < 1$, for $k = 1, 2, \ldots, l-1$, and $\frac{1}{n+1} < 1$. It follows from (3.47) that there is a constant $M_2 > 0$ such that

$$\int_0^T \left|x'(t)\right|^{n+1} dt \leqslant M_2. \tag{3.49}$$

**Case 2.** If $d / \int_0^T |x'(s)|\, ds \leqslant \delta(l)$, then from (3.45) we get

$$\begin{aligned}
\left(d + \int_0^T \left|x'(s)\right| ds\right)^{l-k} &= \left(\int_0^T \left|x'(t)\right| dt\right)^{l-k} \left(1 + \frac{d}{\int_0^T |x'(t)|\, dt}\right)^{l-k} \\
&\leqslant \left(\int_0^T \left|x'(t)\right| dt\right)^{l-k} \left[1 + \frac{(l+1)d}{\int_0^T \left|x'(t)\right| dt}\right] \\
&= \left(\int_0^T \left|x'(t)\right| dt\right)^{l-k} + (l+1)d\left(\int_0^T \left|x'(t)\right| dt\right)^{l-k-1} \\
&\leqslant T^{\frac{n(l-k)}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l-k}{n+1}} \\
&\quad + (l+1)dT^{\frac{n(l-k-1)}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l-k-1}{n+1}},
\end{aligned}$$

which results in

$$\begin{aligned}
&T^{\frac{n-k}{n+1}} \left(d + \int_0^T \left|x'(t)\right| dt\right)^{l-k} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{k+1}{n+1}} \\
&\quad \leqslant T^{\frac{n-k+nl-nk}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l+1}{n+1}} + (l+1)dT^{\frac{nl-nk-k}{n+1}} \left(\int_0^T \left|x'(t)\right|^{n+1} dt\right)^{\frac{l}{n+1}}.
\end{aligned}$$

It follows from (3.47) that

$$\left[\beta - (r+\varepsilon)\right] \int_0^T \left|x_1'(t)\right|^{n+1} dt$$

$$\leqslant \left[2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}} + \sum_{k=1}^{l-1} (s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k+nl-nk}{n+1}}\right] \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{l+1}{n+1}}$$
$$+ \sum_{k=1}^{l-1} (l+1)(s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k d T^{\frac{nl-nk-k}{n+1}} \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{l}{n+1}}$$
$$+ \sum_{k=1}^{l-1} a_{k,\rho} 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{k+1}{n+1}}$$
$$+ \left[p_\rho + \left(\int_0^T |e(t)|^{\frac{n+1}{n}}\,dt\right)^{\frac{n}{n+1}}\right] \cdot \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{1}{n+1}}. \tag{3.50}$$

(1) If condition $[A_1]$ holds, then

$$\left(\beta - (r+\varepsilon) - 2^{\frac{l}{n+1}} s_l |\tau|_0^l T^{\frac{n-l}{n+1}} - \sum_{k=1}^{l-1} (s_k+\varepsilon) 2^{\frac{k}{n+1}} \delta^k T^{n-k}\right) \int_0^T |x'(t)|^{n+1}\,dt$$
$$\leqslant \sum_{k=1}^{l-1} (l+1)(s_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k d T^{\frac{nl-nk-k}{n+1}} \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{l}{n+1}}$$
$$+ \sum_{k=1}^{l-1} a_{k,\rho} 2^{\frac{k}{n+1}} |\tau|_0^k T^{\frac{n-k}{n+1}} \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{k+1}{n+1}}$$
$$+ \left[p_\rho + \left(\int_0^T |e(t)|^{\frac{n+1}{n}}\,dt\right)^{\frac{n}{n+1}}\right] \cdot \left(\int_0^T |x'(t)|^{n+1}\,dt\right)^{\frac{1}{n+1}}. \tag{3.51}$$

By (3.36), we see

$$\beta - (r+\varepsilon) - 2^{\frac{l}{n+1}} r_l |\tau|_0^l T^{\frac{n-l}{n+1}} - \sum_{k=1}^{l-1} (r_k+\varepsilon) 2^{\frac{k}{n+1}} |\tau|_0^k T^{n-k} > 0,$$

and also we have $\frac{l}{n+1} < 1$, $\frac{k+1}{n+1} < 1$ ($k = 1, 2, \ldots, l-1$) and $\frac{1}{n+1} < 1$. So it follows from (3.51) that there is a constant $M_3 > 0$ independent of $\lambda$ such that

$$\int_0^T |x'(t)|^{n+1}\,dt < M_3. \tag{3.52}$$

(2) If condition $[A_2]$ holds, then $\frac{l+1}{n+1} < 1$; and by (3.37), we see $\beta > r + \varepsilon$. Moreover, $\frac{l}{n+1} < 1$, $\frac{k+1}{n+1} < 1$, $k = 1, 2, \ldots, l-1$, and $\frac{1}{n+1} < 1$. It follows from (3.47) that there is constant $M_4 > 0$ independent of $\lambda$ such that

$$\int_0^T \left|x'(t)\right|^{n+1} dt < M_4. \tag{3.53}$$

Let $M = \max\{M_1, M_2, M_3, M_4\}$. Then by (3.48), (3.49), (3.52), and (3.53), we see

$$\int_0^T \left|x'(t)\right|^{n+1} dt < M,$$

i.e., (3.46) holds. Thus,

$$|x|_0 \leqslant d + T^{\frac{n}{n+1}} (M)^{\frac{1}{n+1}} := M_0.$$

The remainder can be proved in the same way as in the proof of Theorem 3.1. $\square$

As an application, let us consider the following Rayleigh equation:

$$x''(t) = f\big(x'(t)\big) + h\big(x(t)\big) + g\big(x\big(t - \tau(t)\big)\big) + e(t), \tag{3.54}$$

where $f, g, h$, and $e$ are continuous on $R$ with $e(t+T) \equiv e(t)$. Moreover, for $\forall c \in R$, $f(0) + g(c) + h(c) + e(t) \not\equiv 0$. If

$$g \in C^l(R, R), \quad \left|\frac{g^{(l)}(x)}{l!}\right| \leqslant s_l, \ \forall x \in R, \quad \text{and}$$

$$\lim_{|x| \to +\infty} \frac{|g^{(k)}(x)|}{k!|x|^{l-k}} \leqslant s_k \quad (k = 1, 2, \ldots, l-1),$$

where $l$ is a positive integer and $s_k \geqslant 0$ $(k = 1, 2, \ldots, l-1)$ is a constant, then $\forall x, y \in R$, we have

$$\begin{aligned}\big|g(x) - g(y)\big| &= \bigg|g'(y)(x-y) + \frac{g''(y)}{2!}(x-y)^2 + \cdots + \frac{g^{(l-1)}(y)}{(l-1)!}(x-y)^{l-1} \\ &\quad + \frac{g^{(l)}(\xi)}{l!}(x-y)^l\bigg| \\ &\leqslant s_l|x-y|^l + \sum_{k=1}^{l-1}\left|\frac{g^{(k)}(y)}{k!}\right||x-y|^k,\end{aligned}$$

where $\xi$ is between $x$ and $y$. So by using Theorem 3.2, we can obtain the following result.

**Corollary 3.1.** *Suppose that* (3.55) *and the following assumptions hold*:

$[H_1']$ *There exists a constant $d > 0$ such that $f(0) + h(x) + g(y) > |e|_0$, for $(x, y) \in R^2$ with $x \geqslant y > d$.*

$[H_2']$ *There exists a constant* $d > 0$ *such that* $f(0) + h(x) + g(y) < -|e|_0$, *for* $(x, y) \in R^2$ *with* $x \leqslant y < -d$. *Furthermore, there is a constant* $\beta > 0$ *such that* $xf(x) \leqslant \beta|x|^{n+1}$, $\forall x \in R$, *where* $n \geqslant 1$ *is a constant.*

*Then Eq.* (3.54) *has a* $T$*-periodic solution if one of the following conditions holds*:

$[A_1']$ $l = n$, *and*

$$2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k} < \beta;$$

$[A_2']$ $l < n$.

**Example 3.1.** Let us consider the following equation:

$$x''(t) = -5\big[x'(t)\big]^3 + 4\big[x(t)\big]^5 - \left[x\left(t - \frac{1}{100}\sin t\right)\right]^3 + \cos t. \tag{3.55}$$

Corresponding to Eq. (3.54), one can find $f(x) = -5x^3$, $h(x) = 4x^5$, $g(x) = x^3$, $\tau(t) = \frac{1}{100}\sin t$, and $e(t) = \cos t$. So $n = l = 3$, $|\tau|_0 = \frac{1}{100}$, $\beta = 5$, and $T = 2\pi$; and by (3.54) we know $s_3 = 1$, $s_2 = 3$, $s_1 = 3$. Thus

$$2^{\frac{l}{n+1}} s_l |\tau|_0^l + \sum_{k=1}^{l-1} 2^{\frac{k}{n+1}} s_k |\tau|_0^k T^{n-k} = \frac{2^{3/4}}{100^3} + \frac{3\pi^2 2^{1/4}}{25} + \frac{6\sqrt{2}\pi}{100^2} < 5 = \beta.$$

Furthermore, from $h(x) = 4x^5$, $g(x) = x^3$, one can easily see that assumptions $[H_1']$ and $[H_2']$ hold. So by applying Corollary 3.1, we know that Eq. (3.56) has at least one $2\pi$-periodic solution.

**Remark 3.1.** From the above example, we see the degrees of the variables $x_0, x_1$, and $x_2$ in functions $f(x_0)$, $g(x_2)$, and $h(x_1)$ are all greater than 1, which is different from growth condition (1.2) and (1.4) assumed by [2] and [3], respectively. Furthermore, the condition imposed on $g(x)$ satisfies (3.55), which is different from (1.6) assumed by [6].

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